Vandermonde determinant*

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A Vandermonde matrix of order n has the form

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ {x_1}^2 & {x_2}^2 & {x_3}^2 & \cdots & {x_n}^2 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ {x_1}^{n-1} & {x_2}^{n-1} & {x_3}^{n-1} & \cdots & {x_n}^{n-1} \end{pmatrix}$$

Although the transpose of this may be used as the definition; and it may also be referred to as an *alternant* matrix. Whilst solving an equation with an $n \times n$ Vandermonde matrix requires $O(n^2)$ operations, it turns out that the determinant is very easy to calculate, using the formula:

$$\prod_{1 \leq i < j \leq n} (x_j - x_i)$$

For example, the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{pmatrix}$$

Or its transpose, has determinant $(y-x)(z-x)(z-y)$. This could be proved by multiplying out this expression and checking that it gives the same result as a row expansion of the above matrix, but a more elegant solution (which also illustrates why the general $n \times n$ result holds) is to use elementary column operations to obtain a lower-triangular matrix, for which the determinant is simply the product of the diagonal.

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$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ x^2 & y^2-x^2 & z^2-x^2 \end{vmatrix}$$

By subtracting column 1 from each of columns 2 and 3. Then factorising the bottom row we get

$$\begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ x^2 & (y-x)(y+x) & (z-x)(z+x) \end{vmatrix}$$

So we can pull out factors of (y-x) and (z-x) by the multilinearity of the determinant function:

$$(y-x)(z-x)\begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 1 \\ x^2 & y+x & z+x \end{vmatrix}$$

Now we perform a further column operation, subtracting the second column from the third, which cancels the 1 and x:

$$(y-x)(z-x)\begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ x^2 & y+x & z-y \end{vmatrix}$$

Again pull a factor forward:

$$(y-x)(z-x)(z-y)\begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ x^2 & y+x & 1 \end{vmatrix}$$

Observe that this is a triangular matrix, so its determinant is the product of the diagonal $= 1 \times 1 \times 1 = 1$. So we are left with simply the factors $(y-x)(z-x)(z-y)$ as desired.