Identity of Sophie Germain*

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$$a^4 + 4b^4 = (a^2 + 2b^2 + 2ab)(a^2 + 2b^2 - 2ab)$$

This innocent looking factorisation is easy to verify (for a 'proof', simply multiply out the right hand side), but very difficult to spot. This is partly because it seems to contradict a good rule of thumb- that whilst an expression of the form $a^2 - b^2$ (the difference of two squares) readily factors, $a^2 + b^2$ does not. Thus following a natural first step in trying to simplify the left hand side of the above equation by rewriting it as $(a^2)2 + (2b^2)^2$ offers no insight.

However, the observation that $a^2 + b^2$ does not factor lacks precision. Over the complex numbers, it *is* the difference of two squares: $a^2 + b^2 = a^2 - -b^2 = a^2 - i^2b^2$ and so it factorises as $(a + ib)(a - ib)$. Generally, however, one tries to preserve properties of the original object when factoring- in particular, when trying to factor a natural number, one hopes for a natural factorisation (this is the challenge of prime factorisation). The Identity of Sophie Germaine is hence of interest since if a and b are natural numbers, the two factors on the right hand side will also be (and hence rational values for a and b give rise to at worst rational factors, and so on). The reason that the Identity of Sophie Germain holds is that $a^2 + b^2$ *will* actually factor neatly, provided $2ab$ is also a perfect square; if $2ab = c^2$ then

$$a^2 + b^2 = (a + b + c)(a + b - c)$$

Knowing how to express an object as a sum usually tells you nothing about how to express it as a product (that is, to find factors); finding clever factorisations is therefore helpful in number theory. For instance, knowing that $91 = 90 + 1$ offers no clues in discovering that $91 = 13 \times 7$, whereas writing $91 = 70 + 21$ *is* enlightening. The above identity ensures that no prime number other than 5 takes the form $a^4 + 4b^4$, and many similar restrictions follow. You may like to try and

*First appeared on Everything2, at *http://www.everything2.com/index.pl?node_id=1796348* verify that $454^5 + 545^4$ cannot be prime[1], or indeed that $n^4 + 4^n$ is never prime for any natural number greater than 1.

As a historical footnote, such were the "customs and prejudices" of 18th century France that Sophie Germain had to hide her true identity (or more precisely, her gender) to engage in mathematics, assuming instead the identity of one Monsieur Le Blanc.

[1] A hint- noting that this multiplies out to 1326463407 2747908684 3491350425 7096463479 9744598942 8178686378 8739968567 4733928143 0539115157 0625738339 0916580334 9158665854 2484250541 3315361299 9565901785 1623898621 6565917389 0530583817 0326578582 1026139489 4145763898 2081327972 2433720660 6966360215 1582332068 4351327863 5939443213 8172266022 3746213949 3631596066 2466647213 3934520391 498014849 is not helpful!