Height*

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In mathematics, a 'height' is sometimes used to describe the akwardness of an object, where the usual notions of size or boundedness are unhelpful in doing so. For instance, the number $\frac{1999}{2000}$ is very close in absolute size to 1, but is much harder to manipulate (quickly- what is $\frac{1999}{2000}$ cubed?); whereas the number 100 is much larger than 1, yet easier to, say, multiply by than $\frac{1999}{2000}$. Thus rather than thinking in terms of size, we may consider the height of a rational number, defined as the maximum of $|p|$ and $|q|$ for a number of the form $\frac{p}{q}$ (in lowest terms). Then the height of 1 is 1, and that of 100 is 100: significantly bigger, but far less than the height of our awkward $\frac{1999}{2000}$, which clocks in at 2000.

This approach has a number of uses. For instance, the height of a rational number as defined above is a non-negative integer, and thus well-ordered: the technique of proof by infinite descent can be used. Another application is in a proof of the countability of the rational numbers: for each natural number n, the set of rational numbers of height n is finite (neither the numerator nor denominator, each an integer, can fall outside the range $-n \ldots n$) and any rational number falls into one of those sets, ensuring that the rationals are a countable union of countable (since finite) sets and thus themselves countable.

Whilst rational numbers provide a gentle introduction to the notion of height, more complicated objects can be tackled in this way. For instance, the height of a polynomial $P = a_0 + a_1 x + \ldots + a_n x^n$ is

$$H(P) = \max\{|a_0|, |a_1|, \ldots, |a_n|\}$$

whilst for an algebraic integer ζ such that the polynomial of smallest degree with ζ as a root is $P = a_0 + a_1 x + \ldots + a_n x^n$, we define the height of ζ by

$$h(\zeta) = n + \sum_{i=0}^{n} |a_i|$$

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Similarly to the rationals, organising the algebraic numbers via height proves that they are countable, and hence that there are transcendental numbers. The height of a polynomial is of interest when considering computational aspects of polynomial algebra. For instance, in the greatest common divisor node I gave an example of two polynomials of height 1 whose gcd had height 2- that is, such a computation can create output which is more complicated than the input, and algorithms cannot therefore depend on a decrease in height despite a probable decrease in degree.

Probably the most powerful application of height is in the study of algebraic number theory, and as a special case elliptic curves. It is possible to give a description of a height function on an abelian group, giving rise to the Mordell-Weil theorem.