General Linear Group*

Graeme Taylor

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The general linear group is a useful entry point into the mathematical realm of group theory (roughly, the abstract study of symmetries), since it ties that subject into the more familiar (for most undergraduates) linear algebra. Translating abstract notions of groups into more recognisable properties of matrices (or, in turn, linear maps) can make it easier to grasp them at first, and provides some indication of the power of such abstraction once it is subsequently applied to other groups (such as those of permutations).

There isn't one general linear group; rather a given example specifies two parameters- the general linear group $GL(n, F)$ (also written $GL_n(F)$) comprising the set of all square matrices $M \in Mat_{n,n}(F)$ with non-zero determinant. For example, $GL(2, \mathbb{R})$ is the set of matrices

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ such that } a, b, c, d \in \mathbb{R}, ad - bc \neq 0 \right\}$$

In other words, the general linear group contains only invertible matrices (since a matrix is invertible if and only if it has non-zero determinant). This restriction is necessary since if we simply take the set of all $n \times n$ matrices, the group axioms do not always hold. With this construction, and the familiar matrix multiplication as the group operation, these axioms can be easily verified:

For $A, B, C \in GL(n, F)$ using matrix multiplication $*$

- *Associativity*: By definition of matrix multiplication, $(A * B) * C = A * (B * C)$.
- *Identity*- $\exists e \in GL(n, F)$ such that $A * e = A = e * A$: This relation is satisfied by the Identity matrix I_n. We can chose this matrix as e since it is of size $n \times n$, has determinant 1 and (as any field contains 0 and 1) is therefore in $GL(n, F)$.

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- *Inverses*: for any $A \in GL(n, F)$, there is $A^{-1} \in Mat_{n,n}(F)$ with $AA^{-1} = I = A^{-1}A$, else A wouldn't be invertible and hence would have zero determinant (contradicts assumption of A being in the general linear group). We have already shown that I is our identity element e for the group; so all we require is that A^{-1} also be in $GL(n, F)$. But if A is $n \times n$ in size, so is A^{-1}; and as $det(A)$ does not equal zero, $det(A^{-1}) = \frac{1}{det(A)}$ by properties of determinant is also non zero. We are done.

Some formulations of the group axioms require that closure be checked- that is, if $A, B \in G$ for some group G, then $A * B$ should also be an element of G. Since our chosen operator, matrix multiplication, is defined on a larger set, that of all matrices, it would be wise to check this condition here to make sure that applying the operation doesn't cause us to 'fall out' of the group. If A and $B \in GL(n, F)$ then $A * B = AB$ is also an $n \times n$ matrix. By the product rule for determinants, $det(A * B) = det(AB) = det(A)det(B)$. This cannot be zero for $det(A)$, $det(B)$ non-zero, so AB satisfies the conditions for membership of $GL(n, F)$.

The notion of the general linear group, besides providing a compact way of refering to the invertible matrices, is a starting point for several relations and groups of interest in linear algebra/group theory. For instance, if V is a vector space over F, the group

$$GL(V) = \{\alpha | \alpha : V \to V \text{ is an invertible linear transformation}\}$$

is isomorphic to $GL(n, F)$.

If the field F is itself finite, then $GL(n, F)$ will be finite for a given n. For instance, with $F_2 = \{0, 1\}$ the group $GL(2, F_2)$ comprises the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

The determinant function can be thought of as a mapping $GL(n, F) \to F^x (= F \backslash \{0\})$, this is then a homomorphism by the product rule for determinants.

When $n > 1$, $GL(n, F)$ is not an abelian group.

The *special linear group* $SL(n, F)$ is the set of $n \times n$ matrices with determinant 1, or $\{g | g \in GL(n, F), det(g) = 1\}$. It is an example of a normal subgroup of $GL(n, F)$, since $\forall h \in SL(n, F)$, $\forall g \in GL(n, F)$ we have $det(ghg^{-1}) = det(g)det(h)det(g^{-1}) = \dfrac{det(g)}{det(g)} = 1$ (by various properties of determinant and $SL(n, F)$ definition).