Methods for visualising fluid flow[*]

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In continuum/fluid mechanics, it is often desirable to be able to visualise the way in which a fluid flows. However, this presents obvious difficulties in that the actual flow is a 4-dimensional structure, tracing through spacetime; and so several techniques have been developed to project this into a 3- or 2-D form. Even with a flow that only varies in two dimensions (as shall be considered here for simplicity), plotting time as a third dimension can cause confusion, so an approach that only requires two dimensions is preferable even at the expense of multiple plots.

First however, a precise notion of what it is that should be graphed is needed. At one level this is obvious- we are interested in the velocity of fluid particles at various points in spacetime. The issue is which of these two is the more relevant- the particles, or the points. Following the progress of a given particle, and defining the velocity as its rate of change of position, is the Lagrangian or material view, and seems the most obvious. Yet although intuitive, it is often difficult to work with in practice, which gives rise to the Eulerian view of motion: fixing a specific point in 3-D space and defining a velocity field at that point at a given time via the velocity of whichever particle is passing through at the chosen time.

Working with this Eulerian view, we have:

- A point $\mathbf{r} = (x, y, z)$

- A velocity field $\mathbf{u} = \mathbf{u}(x, y, z, t)$, the velocity of the particle at $\mathbf{r}$ at time t.

Note that if the $\mathbf{u} = \mathbf{0}$ for a given $\mathbf{r}$, then the point is described as a stagnation point; whereas if the partial derivative with respect to time, $\frac{\partial \mathbf{u}}{\partial t}$ is zero then the flow is steady. These are not necessarily the same; in the first the particles are not moving; whereas for the second all particles that flow through $\mathbf{r}$ have the same velocity (e.g. a waterfall).

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Streamlines

For a fixed time T, a streamline is a line for which the tangent is everywhere parallel to the direction of flow, i.e. $\mathbf{u}(x, y, z, T)$. Since the velocity will take component form $\mathbf{u} = (u_1, u_2, u_3)$ and the tangent will be in the direction $d\mathbf{r} = (dx, dy, dz)$, this gives rise to the condition

$$\frac{dx}{u_1} = \frac{dy}{u_2} = \frac{dz}{u_3}$$

This could be formulated as a pair of coupled ODEs for two of the variables in terms of the third e.g. in terms of x,

$$\frac{dy}{dx} = \frac{u_2(x, y, z, T)}{u_1(x, y, z, T)}$$

$$\frac{dz}{dx} = \frac{u_3(x, y, z, T)}{u_1(x, y, z, T)}$$

Alternatively, a parametrisation could be introduced, such that $\frac{d\mathbf{r}}{d\lambda} = \mathbf{u}$, which gives

$$\frac{dx}{d\lambda} = u_1$$

$$\frac{dy}{d\lambda} = u_2$$

$$\frac{dz}{d\lambda} = u_3$$

Provided the flow is steady, the streamline will describe the path of the fluid elements; otherwise it gives an instantaneous snapshot, and a series of streamline sketches will be needed to indicate this evolution through time.

Example streamline calculation

Consider a simple velocity field described by $\mathbf{u} = (x, -y, 0)$. If the more compact form $\frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{0}$ makes you squeamish (you'll learn not to be too disturbed by a blatant division by zero if you do a lot of these), then the coupled ODE form in x is

$$\frac{dy}{dx} = \frac{-y}{x}$$

$$\frac{dz}{dx} = \frac{0}{x} = 0$$

Which indicates that z is simply a constant, and we have a first order separable differential equation for x and y, by rearrangement, namely

$$\int \frac{dx}{x} = -\int \frac{dy}{y}$$

The solution to this is (as an exercise for the reader) $y = \frac{A}{x}$ for a constant of integration A.

It is rather difficult to sketch such a thing in plaintext. Depending on the sign of A (unknown) we obtain (for a sketch in the xy plane, with z arbitrary) curves in all four quadrants, which are asymptotic to the x and y axes. You can determine the direction of flow on these curves by checking the original velocity field. For instance, in the first quadrant, x and y are both positive, so the first component of u is positive and the second negative, giving a flow downwards to the right.

Particle Paths

The concept of the particle path is in line with the Lagrangian view; we consider the particle at a given point at an initial time and find the curve it traces through the fluid as time increases. For an initial position $\mathbf{d}$ at $t = 0$, and position $\mathbf{r}$ having moved with velocity $\mathbf{u}$ for a time t, we obtain the ODE

$$\frac{d\mathbf{r}}{dt} = \mathbf{u}(x, y, z, t); \mathbf{r}(0) = \mathbf{d}$$

Which by equating components yields this time

$$\frac{dx}{dt} = u_1(x, y, z, t)$$

$$\frac{dz}{dt} = u_2(x, y, z, t)$$

$$\frac{dz}{dt} = u_3(x, y, z, t)$$

Example particle path calculation

Retaining our velocity field $\mathbf{u} = (x, -y, 0)$, let us find the path of the particle which moves through $(1, 1, 0)$ at time $t = 0$. Evaluating each component equation, we get

$$\frac{dx}{dt} = x \text{ so } x = Ae^t, \ A \in \mathbb{R} \text{ a constant}$$

$$\frac{dy}{dt} = -y \text{ so } y = Be^{-t}, \ B \in \mathbb{R} \text{ a constant}$$

$$\frac{dz}{dt} = 0 \text{ so } z = C \in \mathbb{R} \text{ a constant}$$

From initial conditions $x(0) = y(0) = 1$, $z(0) = 0$ we conclude that $A = B = 1$, $C = 0$. Then we can eliminate t to obtain the particle path $y = \frac{1}{x}$. This time we have only a single curve, starting at the point $(1, 1)$ (in the xy plane) and hence restricted to the first quadrant. The direction of flow should be obvious, but can be confirmed from the velocity field as before. This curve belongs to the set of those found for the streamlines due to the chosen veolcity field being steady.

Streaklines

Whilst the particle path tracked the progress of a single particle from a known starting time and position, the streakline represents the status of all particles which passed through a fixed point at any time in the past. This can be observed experimentally by injecting dye into the fluid at a certain point- the spread of dye will be the streakline, and will obviously evolve with time, although should the flow be steady then it will be the same as the particle path.

To graph the streakline for a known velocity field can generate considerably more complicated equations than for steamlines or particle paths. In fact, the general solution found for a particle path is the starting point for a streakline calculation. The system to be solved is

$$\frac{d\mathbf{r}}{dt} = \mathbf{u}(x, y, z, t); \mathbf{r}(\tau) = \mathbf{d}$$

The solution $\mathbf{r}(t)$ is the position at time t; eliminating τ yields the streakline.

Example streakline calculation

Since the chosen velocity field is steady, we need to introduce some time dependency if a distinct result from the above is to be obtained. So let $\mathbf{u} = (\frac{x}{t+1}, y, 0)$. Proceed as for a particle path calculation to find the general solution, but don't plug in the initial conditions- in this case the answer should be

$$x = C(t+1)$$

$$y = Be^t$$

$$z = A$$

We seek the path of any particle which passes through $(1, 1, 0)$ at some time $t = \tau$, so we have $C(\tau + 1) = 1 = Be^\tau$; $A = 0$. These are reinserted into the general solution to give

$$x = \frac{(t+1)}{(\tau+1)}$$

$$y = e^{t-\tau}$$

$$z = 0$$

Now since $\tau = \frac{t+1}{x} - 1$ by rearrangement, we can eliminate it to obtain the rather involved streakline in the xy plane:

$$y = \exp(t - \frac{t+1}{x} + 1) = \exp((t+1)(1 - \frac{1}{x}))$$

Which can be sketched for various t.