Conservative*

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In vector calculus, the line integral of a vector-valued function on some curve is called the work integral along that curve. The curve can be chosen arbitrarily, but this may change the value of the work integral. If it turns out that the amount of work done is the same regardless of the curve chosen, then the function is described as conservative. In physical terms, this means that the effort required to move within a force field F between two points depends on the distance between them, not the route taken: there can be no shortcuts, but there are no 'bad' routes either.

It further transpires that having a conservative function is equivalent to some other interesting properties (based on the various derivatives possible within vector calculus) of vector functions. These shall be illustrated in this writeup; and will hopefully give an insight into some of the key ideas and typical proof methods encountered in first approaching this field of mathematics. For the purpose of clarity functions in $\mathbb{R}^3$ (3-dimensional space, approximately the universe we live in) will be used during further discussion, and the symbol $\cdot$ shall denote the dot product.

First then, let us state the definition of conservative in a more rigorous fashion. There are two commonly used defintions: I'll use the one that corresponds to the idea described above and then later derive the other formulation.

Let C be a curve in $\mathbb{R}^n$, parametrised by a function of a variable t such that $r : \mathbb{R} \Rightarrow \mathbb{R}^n$. So,

$$C = \{\mathbf{r}(t) | t \in [t0, te]\}$$

Then the work integral of a vector field $\mathbf{F} : \mathbb{R}^n \Rightarrow \mathbb{R}^n$ along C is given by

$$\int_C \mathbf{F} \cdot d\mathbf{r} := \int_{t_0}^{t_e} \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt} dt$$

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If C_1 and C_2 are arbitrary curves with the same endpoints (that is, $\mathbf{r}_1(t_0) = \mathbf{r}_2(t_0) = \mathbf{x_0} \in \mathbb{R}^n$, and $\mathbf{r}_1(t_e) = \mathbf{r}_2(t_e) = \mathbf{x_e} \in \mathbb{R}^n$ where $\mathbf{r}_1$ is the parametrisation for C_1 and $\mathbf{r}_2$ is that of C_2) then $\mathbf{F}$ is called conservative iff

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

From here on, our discussion is restricted to $\mathbb{R}^3$, with Ω a simply connected domain

Theorem: If F is conservative, F has a scalar potential.

This is our first interesting result, and we use it to obtain the other defintion of conservative. A scalar potential of $\mathbf{F} : \Omega \Rightarrow \mathbb{R}^3$ is a function Φ satisfying $\nabla\Phi = \mathbf{F}$. The proof would be an absolute massacre in HTML, but here is the method if not the madness:

- We can define a candidate $\Phi(x)$ as precisely the work integral of $\mathbf{F}$ along a curve from the origin $\mathbf{0} = (0, 0, 0)$ to $\mathbf{x} = (x, y, z)$ in Ω; this is a well defined function of $\mathbf{x}$ since the value obtained is consistent regardless of the curve chosen, precisely because $\mathbf{F}$ is conservative.

- If we consider $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ (or other appropriate orthogonal curvilinear coordinate system; I adopt **i,j,k** for clarity here) then we can show that $\frac{\partial\Phi}{\partial x}(\mathbf{x}) = F_1(\mathbf{x})$
(To do so, take $C := C_1 \cup C_2$ where C_1 is the straight line from $\mathbf{0}$ to (0,y,z) whilst C_2 is the straight line from (0,y,z) to $\mathbf{x}$; the integral over C can be split linearly into two integrals, one over each of these curves).

- By Similar methods for $\frac{\partial\Phi}{\partial y}(\mathbf{x})$ and $\frac{\partial\Phi}{\partial z}(\mathbf{x})$ (constructing C_1 and C_2 such that the first goes to (x,0,z) or (x,y,0) respectively) we can conclude that
$\frac{\partial\Phi}{\partial x}(\mathbf{x}) = F_1(\mathbf{x})$; $\frac{\partial\Phi}{\partial y}(\mathbf{x}) = F_2(\mathbf{x})$; $\frac{\partial\Phi}{\partial z}(\mathbf{x}) = F_3(\mathbf{x})$

- But as $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$, it follows that $\mathbf{F} = \nabla\Phi$ - our constructed work integral is a scalar potential as desired.

Theorem: F has a scalar potential iff F is conservative

We have shown that a conservative field has a scalar potential: now we seek to show the converse- having a scalar potential leads to being conservative. Again, a sketch proof is provided. We assume $\Phi : \mathbb{R}^3 \Rightarrow \mathbb{R}$ and let $F = \nabla\Phi$.

- By the chain rule, we can show that $\nabla\Phi \cdot \frac{d\mathbf{r}}{dt} = \frac{d}{dt}(\Phi \circ \mathbf{r})$ (derivative of the composition of Φ with $\mathbf{r}$)

- Hence the defintion of work integral can be re-written as
$\int_C \mathbf{F} \cdot d\mathbf{r} := \int_{t_0}^{t_e} (\frac{d}{dt}(\Phi \circ \mathbf{r}))dt$

- By the fundamental theorem of calculus, this is simply $\Phi(\mathbf{r}(t_e)) - \Phi(\mathbf{r}(t_0))$. It follows that **F** is conservative since the value of the work integral can be determined by knowledge of the end points only, so the route taken between them is irrelevant.

Alternative defintion of conservative

The above result is slightly more powerful than just giving us the property of conservatism. If we consider an arbitrary closed curve in $\mathbb{R}^3$ - that is, one where the start- and endpoints are the same, then we observe that $\mathbf{r}(t_e) = \mathbf{r}(\mathrm{t}_0)$. So given that the work integral is obtained by $\Phi(\mathbf{r}(t_e)) - \Phi(\mathbf{r}(t_0))$, it is obvious that this quantity is zero. So a second common definition of conservative is:

$$\mathbf{F} \text{ is conservative} \Leftrightarrow \int_C \mathbf{F} \cdot d\mathbf{r} = 0 \text{ for all closed curves C.}$$

Theorem: F is conservative iff F is irrotational

Equipped with this new definition, we can easily prove another equivalence, between conservative fields and irrotational ones (having zero curl).

The forward implication is virtually a one-liner: curl F is $\nabla \wedge \mathbf{F}$ (cross product), but since **F** is conservative it has a scalar potential, so we obtain curl F $= \nabla \wedge \nabla\Phi$ for some Φ. Multiplying out you get curl F $= 0$.

The reverse implication is also fairly simple once you are familiar with Stokes' Theorem: by which the integral over a closed curve can be rewritten as a double integral over the surface of which the curve is a boundary. So $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_s (\nabla \wedge \mathbf{F}) \cdot d\mathbf{s}$ for a closed C. Since **F** is irrotational, this is just $\iint 0 \cdot d\mathbf{s} = 0$. By our alternative definition, as C is closed, we have **F** conservative.

The simply connected condition

Swap[1] points out the importance of the domain Ω being simply connected: consider "*$F(x, y, z) = (\frac{-y}{r^2}, \frac{x}{r^2}, 0)$, with $r = x^2 + y^2$. Then curl $F = 0$, but any line integral that encloses the origin evaluates to 2π (the example comes from complex analysis and the calculus of residues. Thus, this field is not conservative, and there is no scalar potential function either.*"

[1]An Everything2 user, *http://www.everything2.com/index.pl?node_id=1394568*

Summary

For a function F in a simply connected domain,

F has a scalar potential $\Leftrightarrow$ F is conservative $\Leftrightarrow$ F is irrotational