# The Baire category theorem and cardinality *

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The Baire category theorem is an example of the power of links between often disconnected areas of mathematical study. More than just the product of the artificial way in which the subject is chopped up (especially to meet the needs of modular university courses), such connections often point to truly fundamental ideas in the subject (such as the beautiful interplay between $\pi$ and $e$, from geometry to algebra to analysis....)

More to the point, such connections can make for elegant or simpler proofs. Often a difficult problem can be mapped to another framework in which it is easier to solve, then the solution translated back to your original area. The Baire category theorem allows for knowledge of cardinality (the size of sets) to be used as a tool in asking questions about complete metric spaces (a particularly well-behaved type of space such as the real numbers), or vice versa.

Instead of the usual formulation of the Baire category theorem as set out in its writeup, I'll make use of a slightly different version:

> **Baire category theorem**: A complete metric space $X$ cannot be written as a countable union of nowhere dense sets. In fact, the complement of such a union is dense.

(This is in effect arguing that any complete metric space is of second category. Hence the slightly misleading name of the theorem, especially when encountered in its usual form- it has nothing to do with category theory!)

So how can we make use of this beyond its usual (and powerful) application to analysis?

## Proof that the set of real numbers is of uncountable size

The usual proof of uncountability is a diagonal argument, but many people find these unintuitive. The Baire category theorem proof runs as follows. Suppose $\mathbb{R}$ is countable. For any $x \in \mathbb{R}$, the

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set $\{x\}$ is closed and has empty interior, so is nowhere dense in $\mathbb{R}$. Yet the union of these nowhere dense sets is the entirety of $\mathbb{R}$. So we have written $\mathbb{R}$ as the union of countably many nowhere dense sets- hence by the Baire category theorem $\mathbb{R}$ is not complete. Yet $\mathbb{R}$ *is* complete with metric $d(x, y) = |x - y|$, so we have a contradiction. Our assumption of $\mathbb{R}$ being countable is therefore false, meaning $\mathbb{R}$ is uncountable.

### Proof that the rationals are incomplete

Here we make use of the Baire category theorem in the opposite way- using cardinality to prove something about completeness (or lack of). Consider the rational numbers $\mathbb{Q}$ again with the standard metric $d(x, y) = |x - y|$. A direct proof of the incompleteness of the metric space $(Q, d)$ would rely upon the defintions- to be complete, a space needs to contain the limit of all its Cauchy sequences. Thus to prove incompleteness here, you'd need to construct a Cauchy sequence of rationals converging to a favourite irrational such as $\sqrt{2}$, a limit which would be missing from the space. Shockingly for a would-be mathematician, I don't have a sequence of rational approximations of root 2 to hand, so I can't proceed in this manner[1]. But I do know the Baire category theorem, and from that the proof is simple- $\mathbb{Q}$ is countable, so the collection $\{\{q\}|q \in \mathbb{Q}\}$ is a countable collection of nowhere dense sets. Its union is clearly $\mathbb{Q}$, so the resulting metric space $(Q, d)$ cannot be dense, and so I'm home.

Of course, these are fairly simple examples, and a complicated proof using this theorem may well require mastery of both cardinality and completeness. Still, it is illustrative of a helpful guiding principle across mathematics- if you can't solve a problem, change the question! From everyday use of Laplace or Fourier transforms to turn differential equations into simpler systems; to the tying together of elliptic curves and number theory in the Taniyama-Shimura conjecture that allowed Wiles' proof of Fermat's Last Theorem, this search for connections in mathematics can be seen as both a practical and beautiful part of the subject.

[1] Swap (An Everything2 User, *http://wwww.everything2.com/index.pl?node_id=1394568*) to the rescue! He offers the sequence generated by iterating Newton's Method for $x^2 - 2 = 0$. Starting out with $x_0 = 1$, define $x_{n+1}$ as $x_n - \frac{x_n^2 - 2}{2x_n}$. These will always be rational, and converge to the root of 2 as desired.