Intermediate Value Theorem*

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There are at least three ways in which the Intermediate Value Theorem can be stated. As you would hope and expect, they all mean the same thing (more on that later), so your choice of formulation will tend to be based on what you intend to apply the function to- and there are many such applications, the IVT being a major tool in analysis.

Versions of the Intermediate Value Theorem

Version 1 *(general version)*

Let $I \subset \mathbb{R}$ be an interval, and $f : I \to \mathbb{R}$ be a continuous function.
Suppose x and y are in I, with $f(x) < f(y)$, and take $c \in \mathbb{R}$ such that $f(x) < c < f(y)$.

Then, $\exists z \in (x, y) \cup (y, x)$ such that $f(z) = c$.

Version 2 *(axis-crossing version, as given by Noether[1] above)*

Let $I \subset \mathbb{R}$ be an interval, and $f : I \to \mathbb{R}$ be a continuous function.
Suppose x and y are in I, with $f(x) < 0 < f(y)$.

Then, $\exists z \in (x, y) \cup (y, x)$ such that $f(z) = 0$.

Version 3 *(Elegant version!)*

Let $f : I \to \mathbb{R}$ be defined on an interval I. Then if $J \subseteq I$ is any interval, $f(J)$ is an interval.

*First appeared on Everything2, at *http://www.everything2.com/index.pl?node_id=1541686*

[1] An everything2 user, *http://www.everything2.com/index.pl?node=noether*

Notes, and a non-mathematical explanation

You may have noticed that the first two versions seem to have generated a lot more mathematical symbols than Noether's version above. This is because the starting point chosen here is a general interval I, rather that a particular one (such as the closed interval $[a, b]$). In this case, knowing $f(a) < f(b)$ doesn't tell you that $a < b$ (consider the function $f(x) = -x$ for a counterexample), so when claiming the existence of z, we have to make sure it falls into a set that isn't empty! The expression $(x, y) \cup (y, x)$ really means (x, y) or (y, x), whichever of these makes sense- if $x < y$, (x, y) is a sensible concept and that's where we find z; otherwise it's in (y, x) as y is the smaller of the two numbers.

The third version does away with all this complication of symbols and is probably the easiest way to get your head around what the IVT is actually saying (that is, the meaning rather than the mathematical definition). Of course, this is only possible if you have an idea of what an interval is. Mathematically, an interval I satisfies $x, y \in I, x < c < y \Rightarrow c \in I$. What this means is that, if I supply you with any two points in the set, you also get every point in between them for free.

Given this, the third version of the IVT says that if you start with this property, and apply a continuous function, it's still true for the new set of values you have- each of which is the function applied to something. Hence if you know the value of the function evaluated at two points, and take a number between those evaluations, then there was something in your original set which, if you applied the function to, would give your chosen point.

As drawing graphs is a bit tricky in ascii, consider if you would the following analogy, in place of drawing lines on paper. Suppose you are climbing a mountain, and you start the climb at a certain height at your start time. After a length of time, you get to a new height. What the IVT says is that if your path was continuous (you couldn't teleport part way up the mountain), then for any height between where you started and where you wound up, there was at least one point in time at which you were at that height. There may in fact be several such points; that point could have been sea level (hence the zero-case, version 2) or you could have started at the top of the mountain and made your way down (hence the ambiguity over $(x, y) \cup (y, x)$), but whatever the setup, you moved through all intermediate values on your journey. So the idea is reasonable and with a bit of analytical rigour (see the proof) we can confirm it in a mathematical fashion (rather than taking it as a scientific hypothesis in the light of compelling evidence such as this real-world comparison).

Book-keeping: Equivalence of the three forms

Feel free to ignore this part if you're just after the definition or head over to the proof to see why any of the versions hold; here I present proof of the equivalence of each formulation which would mostly be needed only if (like me) you've been asked to do so as part of a course of study, probably at a first or second year undergraduate level. As such, I'll pitch the arguments there, with less explanation of what everything means.

Version 1 implies Version 2

Probably the easiest proof- simply set $c = 0$ in the statement of version 1, and you get version 2!

Version 2 implies Version 1

Define a function $h(x) = f(x) - c$. Then, by the second version, there is z such that $h(z) = 0$. Then $f(z) - c = 0$, so $f(z) = c$. This proves the equivalence of 1 and 2; hence to show equivalence of all three we need only show $3 \Leftrightarrow 1$ or $3 \Leftrightarrow 2$.

Version 3 implies Version 1

Suppose we have $f(x) < c < f(y)$. Define an interval J by $[x, y]$ if $x < y$ or $[y, x]$ otherwise. We know $x, y \in J$, so $f(x), f(y) \in f(J)$. Since $f(J)$ an interval by version 3, it follows that if $f(x) < c < f(y), c \in f(J)$. But definition of $c \in f(J)$ means that $\exists z \in J$ st $f(z) = c$ as desired.

Version 1 implies Version 3

Pick interval $J \subset I$, $f(x), f(y) \in f(J)$ with $f(x) < f(y)$. Our first version gives $\exists z \in (x, y) \cup (y, x)$ such that $f(z) = c$. But $(x, y) \cup (y, x) \subset J$, which is an interval, so $z \in J$. So then $c = f(z)$ for some $z \in J$ i.e. $c \in f(J)$ as desired.