Brachistochrone*

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The brachistochrone problem, as so named by Johannes Bernoulli in the 17th century from the greek $\beta\rho\alpha\chi\iota\sigma\tau o\varsigma$ brachistos (shortest) and $\chi\rho o\nu o\varsigma$ chronos (time), is a motivating example for the calculus of variations, which happens to fall into a special class that makes it easier to solve.

Geometrically, it can be described as determining a path from a point P vertically higher than, but not directly above, a second point Q, such that the time of travel for a particle to move under the action of gravity is minimised. Friction is ignored and the particle is assumed to have zero initial velocity (that is, to start at rest at P). The curve described turns out to be neither of the obvious choices- a straight line or circular arc; but as the following analysis will reveal, a cycloid.

Setup

First, we formulate the problem in terms of variational calculus. Working in a vertical plane with cartesian coordinates, take the initial point P as the origin (0,0) and Q to be an arbitrary point (a,b). Then a curve Γ joining those points graphs some function that maps $x \to y(x)$, with $x \in [0, a]$ such that $y(0) = 0$ and $y(a) = b$.

Then, any point on Γ is of the form $(x, y(x))$ and has some velocity, v. By conservation of energy,

$$\frac{(mv^2)}{2} = mgy(x)$$

for m the mass of the particle in motion, g the acceleration due to gravity.

From the formula for the arc length $s(x)$ of Γ from (0,0) to $(x, y(x))$, we can conclude

$$\frac{ds}{dx}(x) = \sqrt{1 + y'(x)^2}$$

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So the time T to traverse the curve Γ is given by

$$\int_0^a \frac{dt}{ds}\frac{ds}{dx}dx = \frac{1}{\sqrt{2g}}\int_0^a \sqrt{\frac{1+y'(x)^2}{y(x)}}dx$$

Dropping the unimportant factor of $\frac{1}{\sqrt{2g}}$, the problem becomes to minimise the integral functional

$$y \to J(y) := \int F(x, u(x), u'(x))dx \text{ between } 0 \text{ and } a,$$
$$\text{where } F(x, y, z) = \sqrt{\frac{1+z^2}{y}}$$

That is, the problem is now in standard form for the calculus of variations.

A special case of the Euler-Lagrange equation.

Observe that in this instance, the functional J is described by a function F which does not depend on its first argument. It follows from the Euler-Lagrange equation that for any extremal u, the function

$$E(x) := F(u(x), u'(x)) - u'(x)\frac{\partial F}{\partial z}(u(x), u'(x))$$

is simply a constant, and so it becomes possible to describe u' in terms of u and some constants of integration. Separation of variables then gives x in terms of u and those constants.

Further, when $F(y, z)$ takes the form $h(y)\sqrt{1+z^2}$, this process is particularly simple; we get, for constants c, d

$$x = \int \frac{du}{\sqrt{\left(\frac{h(u)}{c}\right)^2 - 1}} + d$$

0.1 Application to the Brachistochrone problem

Armed with the above, we see that the Brachistochrone problem satisfies $F(y, z)$ takes the form $h(y)\sqrt{1+z^2}$ with $h(y) = \frac{1}{\sqrt{y}}$. Making a change of variables $u := \frac{\tau}{c^2}$ we obtain

$$x - d = \frac{1}{c^2}\int \sqrt{\frac{t}{1-t}}dt$$

This integral can be solved by a second change of variables; introduce θ such that $\tau := \sin(\frac{\theta}{2})^2$ then by some simple trigonometric identities, the above formula becomes

$$x - d = \frac{1}{2c^2} \int (1 - \cos\theta)\, d\theta$$

Calling the constant $\frac{1}{2c^2}$ A, and remembering that the second argument $y = u(x)$, we have a cycloid

$$x = A(\theta - \sin\theta) + d$$
$$y = A(1 - \cos\theta)$$

as desired.