Analytic Proof of the Baire category theorem*

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I offer here the sort of proof of the Baire category theorem that you might encounter in an undergraduate analysis course (for me, it's a fourth year masters course but its inclusion in the syllabus is fairly arbitrary so other institutions may vary the level at which it appears). For a thought provoking approach via game theory, see ariels'[1] writeup[2]. The Game-theoretic approach is a lot less ugly than this, but obviously requires you to be conversant in game theory (which I'm not).

> **Baire Category Theorem**: For a complete metric space X, and a (countable) sequence of dense open subsets $\{G_n\} \subset X$, the intersection of the G_n is also dense.

First, some notation issues. For the sake of readability, I'll use $\mathbf{G}$ to denote the intersection in question, $\bigcap_{n=1}^{\infty} G_n$, since there will be all manner of awkward subscripts as it is.

Second, the notion of a ball is vital to this proof. Since conventions vary, it's useful to be completely explicit- I'll use what can be referred to as the open ball about a point as follows:

$$B_\delta(x) = \{y \in X \mid d(x, y) < \delta\}$$

denotes a ball of radius δ centred on the point x in X. The presence of strict inequality here renders this an open set.

Proof of the Baire Category Theorem

Let x_0 be a point of X and $\delta_0 > 0$ be arbitrary. Then we seek to show that $B_{\delta_0}(x_0) \cap \mathbf{G}$ is not the empty set (where $B_{\delta_0}(x_0)$ is the ball of radius δ_0 about x_0). That is to say, we can get arbitrarily close to any point we like in X and there is still a point of $\mathbf{G}$ that close (or closer), i.e. $\mathbf{G}$ is dense throughout the space X. We proceed initially by an inductive argument.

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*First appeared on Everything2, at *http://www.everything2.com/index.pl?node_id=1692014*

[1] An Everything2 user, *http://www.everything2.com/index.pl?node=ariels*

[2] See *http://www.everything2.com/index.pl?node_id=456470*

Starting with G_1, we can find an $x_1 \in G_1 \cap B_{\delta_0}(x_0)$. As the finite intersection of open sets (G_1 is open by definition, and any ball is open), $G_1 \cap B_{\delta_0}(x_0)$ is open and hence a neighbourhood of all its points. Since x_1 is such a point, we can construct a ball around it of some radius such that the ball is entirely contained in $G_1 \cap B_{\delta_0}(x_0)$.
For the purposes of the proof, we seek $\delta_1 > 0$ st $B_{2\delta_1}(x_1) \subset G_1 \cap B_{\delta_0}(x_0)$ and $\delta_1 < \dfrac{\delta_0}{2}$.

Now we consider the next dense set G_2. Since it is dense, we can find an x_2 st $x_2 \in G_2 \cap B_{\delta_1}(x1)$. Once again, this is an open set, so we find our next δ_2, satisfying this time $\delta_2 > 0$ st $B_{2\delta_2}(x_2) \subset\ G_2 \cap B_{\delta_1}(x_1)$ and $\delta_2 < \dfrac{\delta_1}{2}$.

In this way the balls are inductively chosen to be nested within one another: in general

$$B_{2\delta_n}(x_n) \subset B_{\delta_{n-1}}(x_{n-1}) \subset B_{2\delta_{n-1}}(x_{n-1}) \subset B_{\delta_{n-2}}(x_{n-2}) \subset \ldots \subset B_{2\delta_1}(x_1) \subset B_{\delta_0}(x_0)$$

So if $n > m$, $B_{2\delta_n}(x_n) \subset B_{\delta_m}(x_m) \subset B_{\delta_0}(x_0)$. Hence $d(x_m, x_n) < \delta_m$ (since x_n lies in a ball of radis δ_m about x_m), which is in turn less than $\dfrac{\delta_0}{2^m}$. †

So our sequence of $\{x_n\}$ is Cauchy and we can appeal to the completeness of X to find $x_\infty \in X$ to which the sequence converges. So fix m and let n tend to infinity. By † we have $d(x_\infty, x_m) \leq\ \delta_m <\ 2\delta_m$. But this means

$$x_\infty \in B_{2\delta_m}(x_m) \subset G_m \cap B_{\delta_{m-1}}(x_{m-1}) \subset G_m \cap B_{\delta_0}(x_0).$$

But m was arbitrary so by the left- and right-most terms of the above expression,

$$x_\infty \in \bigcap_{m=1}^{\infty}(G_m \cap B_{\delta_0}(x_0))$$

That is (by rearrangement of the intersections), x_∞ is an element of $B_{\delta_0}(x_0) \cap \mathbf{G}$, so that set cannot be empty (it has something in it!). We are done. †